In The Name of Allah

Digital Media Laboratory
Advanced Information & Communication Technology Center
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Stochastic Process

Random Walk

Lecture #10

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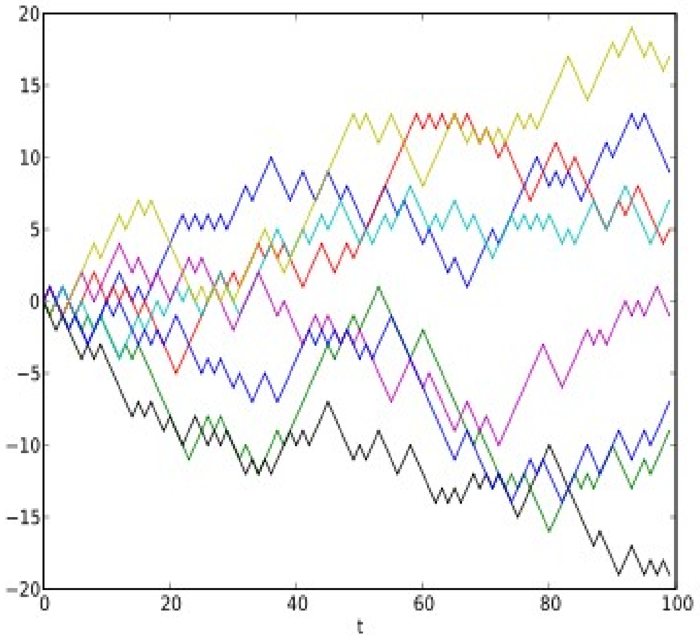
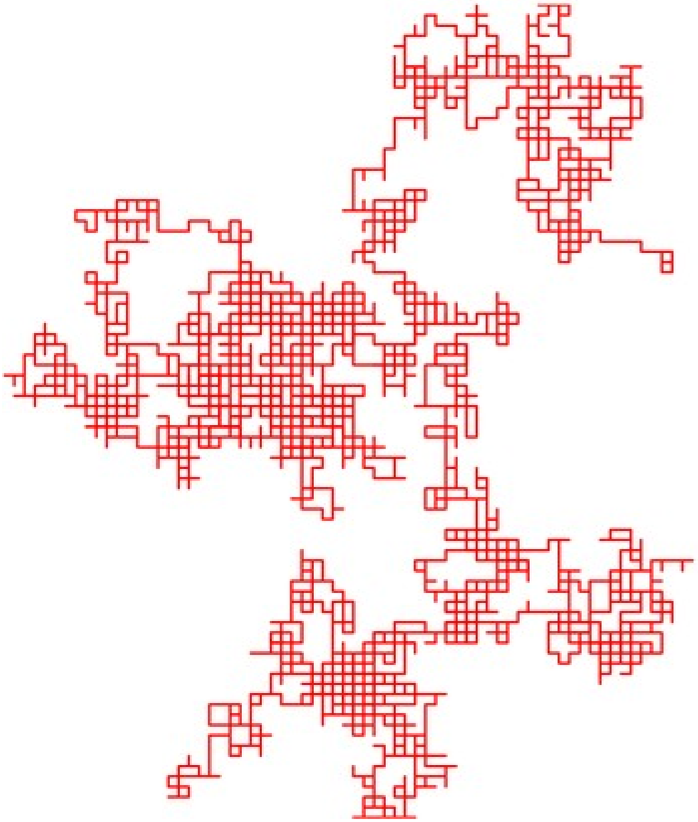
Introduction

- Random walk is a mathematical formalisation of a trajectory that consists of taking successive random steps.
- The results of random walk analysis have been applied to computer science, physics, ecology, economics, psychology and a number of other fields as a fundamental model for random processes in time.
  - the path traced by a molecule as it travels in a liquid or a gas
  - the search path of a foraging animal
  - the price of a fluctuating stock
  - financial status of a gambler
- Some random walks are on graphs, others on the line, in the plane, or in higher dimensions.
- Random walks also vary with regard to the time parameter.
  - Often, the walk is in discrete time, and indexed by the natural numbers, as in $S_0, S_1, S_2, \ldots$
- We focus on one-dimensional Random Walks.

Random Walks in one and two dimensions





Random Walk

- Consider a sequence X of independent random variables that assume values +1 and -1 with probabilities p and q=1-p, respectively. Or equivalently X is and IID Bernoulli random process with parameter p in each time, let S_n denote the partial sum

$$S_n = X_1 + X_2 + \dots + X_n \qquad S_0 = 0$$

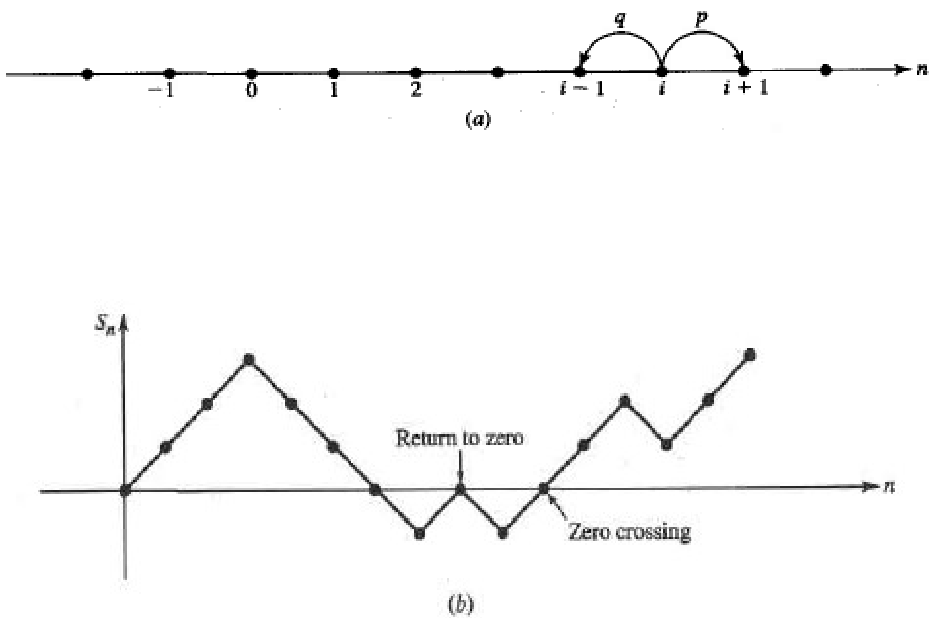
  that represent the accumulated positive or negative excess at the n^{th} trial.

- The random walk is said to be symmetric if p=q=1/2 and unsymmetric if p != q

- Events of special interest and their probabilities:
  - Return to the origin
  - The first return to the origin
  - The r th return to the origin
  - Waiting time for the first gain (first visit to +1)
  - First passage though r > 0

- Exercise: compute mean and variance and autocorrelation of RW process.

Illustration



Recall: Generating Functions

- (Moment) Generating Function of a the sequence a_i is defined as

$$G(s) = \sum_{i=0}^{\infty} a_i s^i$$

For those values of s for which the sum converges.

- probability generating function (PGF) of the random variable X is defined as

$$G_X(s) = \sum_{k=0}^{\infty} p_k s^k = \mathrm{E}(s^X).$$

where

$$p_k = \mathrm{P}(X = k), \qquad k = 0, 1, 2, \ldots$$

- Note that $G_X(1) = 1$ and the series converges for $|s| < 1$. Also $G_X(0) = 0$.

Generating Functions for Common distributions

Bernoulli r.v. – if $p_1 = p$, $p_0 = 1 - p = q$, $p_k = 0$, $k \neq 0$ or 1, then	
$G_X(s) = \mathrm{E}(s^X) = q + ps.$	
Geometric r.v. – if $p_k = pq^{k-1}$, $k = 1, 2, \ldots;$ $q = 1 - p$, then	
$G_X(s) = \dfrac{ps}{1 - qs}$ if $\lvert s \rvert < q^{-1}$	(see HW Sheet 4.).
Binomial r.v. – if $X \sim \mathrm{Bin}(n, p)$, then	
$G_X(s) = (q + ps)^n$, $(q = 1 - p)$	(see HW Sheet 4.).
Poisson r.v. – if $X \sim \mathrm{Poisson}(\lambda)$, then	
$G_X(s) = \sum_{k=0}^{\infty} \dfrac{1}{k!} \lambda^k e^{-\lambda} s^k = e^{\lambda(s-1)}.$	
Negative binomial r.v. – if $X \sim \mathrm{NegBin}(n, p)$, then	
$G_X(s) = \sum_{k=0}^{\infty} \binom{k-1}{n-1} p^n q^{k-n} s^k = \left(\dfrac{ps}{1 - qs}\right)^n$	if $\lvert s \rvert < q^{-1}$ and $p + q = 1$.

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A simple problem

- Let $\{S_n = r\}$ represent the event "at stage n, the particle is at the point r" and $p_{n,r}$ its probability

$$p_{n,r} \triangleq P\{s_n = r\} = \binom{n}{k} p^k q^{n-k}$$

where k represent the number of successes in n trials and n-k the number of failures.

Thus the gain is r = k – (n-k) = 2k-n or k = (n+r)/2

$$p_{n,r} = \binom{n}{(n+r)/2} p^{(n+r)/2} q^{(n-r)/2}$$

note that n and r must have the same parity to have nonzero probability.

Return to the origin

Return to the origin will occur after even number of steps

$$P\{s_{2n}=0\} = \binom{2n}{n}(pq)^n \triangleq u_{2n}$$

$$\begin{aligned} u_{2n} &= \frac{(2n)!}{n!n!}(pq)^n \\ &= \frac{2n(2n-2)\cdots 4\cdot 2}{n!}\cdot\frac{(2n-1)(2n-3)\cdots 3\cdot 1}{n!}(pq)^n \\ &= \frac{2^n n!}{n!}\cdot\frac{2^n(-1)^n(-1/2)(-3/2)\cdots(-1/2-(n-1))}{n!}(pq)^n \\ &= (-1)^n\binom{-1/2}{n}(4pq)^n \end{aligned}$$

So the moment generating function of the sequence $\{u_{2n}\}$ is given by

$$U(z) = \sum_{n=0}^{\infty} u_{2n} z^{2n} = \sum_{n=0}^{\infty}\binom{-1/2}{n}(-4pqz)^n = \frac{1}{\sqrt{1-4pqz^2}}$$

Since U(1) != 1, ... ion. Also since U(1) = ∞ return to origin will occur infinitely often.

First return to the origin

Let V_{2n} denote the probability of this event:

$$B_n = \{s_1 \neq 0, s_2 \neq 0, \ldots, s_{2n-1} \neq 0, s_{2n} = 0\}$$

$$v_{2n} = P(B_n) = P\{s_1 \neq 0, s_2 \neq 0, \ldots, s_{2n-1} \neq 0, s_{2n} = 0\}$$

A visit to the origin at stage 2n is either the first return with probability v_{2n}, or the first return

occurs at an earlier stage 2k<2n with probability v_{2k}, and it is followed b an independent

new return to zero after 2n-2k stages with probability u_{2n-2k}, for k=1,2,...,n. Thus;

$$u_{2n} = v_{2n} + v_{2n-2}u_2 + \cdots + v_2 u_{2n-2} = \sum_{k=1}^{n} v_{2k}\, u_{2n-2k} \qquad n \geq 1$$

$$U(z) = 1 + \sum_{n=1}^{\infty} u_{2n}\, z^{2n} = 1 + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} v_{2k}\, u_{2n-2k}\right) z^{2n}$$

$$= 1 + \sum_{m=0}^{\infty} u_{2m}\, z^{2m} \cdot \sum_{k=0}^{\infty} v_{2k}\, z^{2k} = 1 + U(z)V(z)$$

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First return to the origin (cont.)

$$U(z) = \frac{1}{1 - V(z)}$$

So we would compute V form previously computed U

$$V(z) = \sum_{n=0}^{\infty} v_{2n} z^{2n} = 1 - \frac{1}{U(z)} = 1 - \sqrt{1 - 4pqz^2}$$

$$\begin{aligned}
v_{2n} &= (-1)^{n-1} \binom{1/2}{n} (4pq)^n \\
&= \frac{(-1)^{n-1}(1/2)(-1/2)(-3/2)\cdots(3/2-n)}{n!}(4pq)^n \\
&= \frac{(2n-3)(2n-5)\cdots 3 \cdot 1}{2^n\, n!}(4pq)^n \\
&= \frac{(2n-2)!}{2^{n-1}2^n \cdot n!\,(n-1)!}(4pq)^n \\
&= \frac{1}{2n-1}\binom{2n-1}{n} 2(pq)^n \qquad n \geq 1
\end{aligned}$$

First return to the origin (cont.)

the probability that the particle will sooner or later return to the origin:

$$P\left\{\begin{matrix} \text{particle will ever} \\ \text{return to the origin} \end{matrix}\right\} = \sum_{n=0}^{\infty} P(B_n) = \sum_{n=0}^{\infty} v_{2n}$$

$$= V(1) = 1 - \sqrt{1-4pq}$$

$$= 1 - |p-q| = \begin{cases} 1 - |p-q| < 1 & p \neq q \\ 1 & p = q \end{cases}$$

If p=q=1/2 then with probability l the particle will return to the origin.

The expected value of this random variable is given by

$$\mu = V'(1) = \begin{cases} \dfrac{4pq}{|p-q|} & p \neq q \\ \infty & p = q \end{cases}$$